Two Explorations involving Cylindrical Tanks

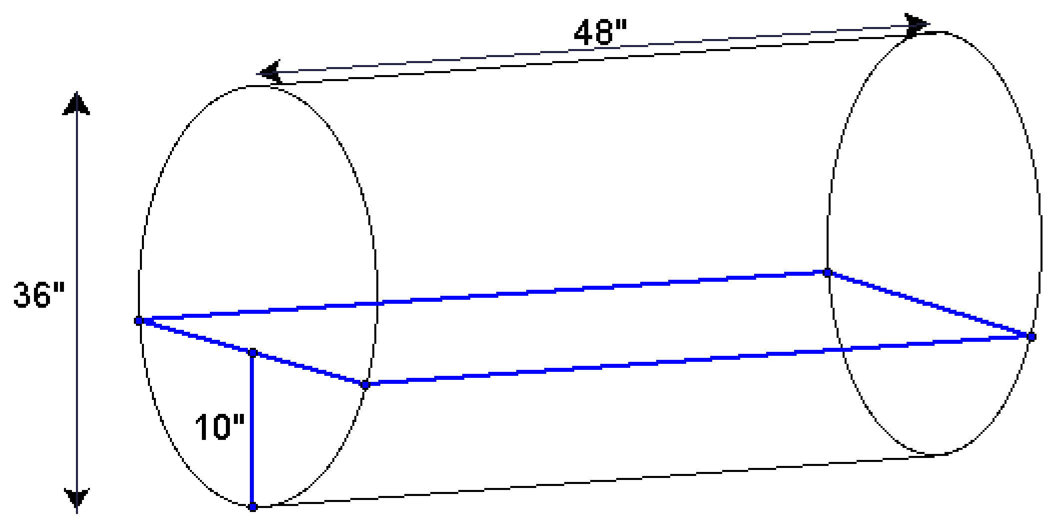
The following two problems were inspired by my work in Dr. Wilson's EMAT 6600 class. There we had to solve a problem to see if a person has enough oil left in his cylindrical shaped tank to last until the end of the winter. Here, we will solve two contextual situations involving cylindrical tanks.

For both problems, first we will try to formulate a general equation then try to make recommendations based on our equation.

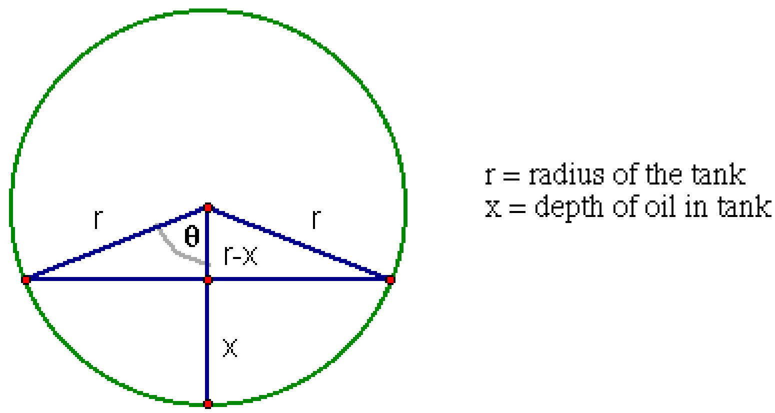
First Problem:

(From Dr. Wilson EMAT 6600 class page)

At the Superintendent's home the furnace used fuel oil stored in an underground tank. It was known that the tank was installed level on its side and that it was 36 inches in diameter and 48 inches long. Using a stick dipped through the fill tube, the superintendent determined he had 10 inches of oil in the tank. He really did NOT want to know how to calculate the amount of oil. He knew from experience that it was February and he would need about 40 gallons of oil to finish the season. Would he have enough oil?



For this somewhat complex diagram, one can simply answer by estimating. For example, if we treated the tank as a cube, then with 48" X 36" X 10" dimension our volume would be 17280 in^3. Since 1 gallon = 231 in^3, we have approximately 75 gallons of oil capacity for the tank. On the other hand, if we approximate the volume of the tank as a triangular prism with altitude = 10”, base = 36”, and length = 48”, the volume would be half of the cubic estimate (i.e. 37.5 gallons). This estimate is pretty close. We, however, need to come up with a precise formula to answer this question and use it for future recommendations.



In order to find the volume, we would have to find the area of the section covered by the oil at the end of the tank and then multiply by the length of the tank. But, how do you find the area of such a figure? Let's begin by examining the end view of the tank (in general so that we can do it for any size cylindrical tank in the future).

First, let’s find the area of the sector formed by the radii to the places where the oil meets the side of the tank. Then we will subtract the area of the triangle from this area of the sector.

Using Pythagorean formula,

$$base_{tank} = 2 \cdot \sqrt{r^2 - (r-x)^2}$$

$$= 2 \cdot \sqrt{r^2 - r^2 + 2rx - x^2}$$

$$= 2 \cdot \sqrt{2rx - x^2}$$

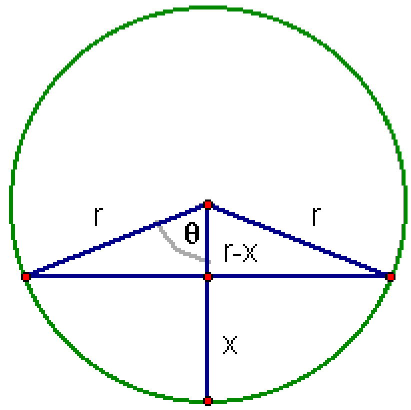
Now, the area of the triangle

$$area_{triangle} = \frac{1}{2} base \times height$$

$$= \frac{1}{2}\left(2\sqrt{2rx - x^2}\right)(r-x)$$

$$= (r-x)\left(\sqrt{2rx - x^2}\right)$$

We also need the area of the measure of the central angle in order to find the area of the sector. Using trigonometric identity,



$$\cos(\theta) = \frac{r-x}{r}$$

$$\theta = cos^{-1}\left(\frac{r-x}{r}\right)$$

For our diagram, the central angle is 2θ. So, the area of the sector can be calculated using the following proportion:

$$\frac{2\theta}{2\pi} = \frac{area_{sector}}{\pi r^2}$$

$$\frac{2cos^{-1}\left(\frac{r-x}{r}\right)}{2\pi} = \frac{area_{sector}}{\pi r^2}$$

$$area_{sector} = \frac{\pi r^2 \cdot 2cos^{-1}\left(\frac{r-x}{r}\right)}{2\pi}$$

$$area_{sector} = r^2 cos^{-1}\left(\frac{r-x}{r}\right)$$

So, area of sector – area of triangle gives us,

$$r^2 cos^{-1}\left(\frac{r-x}{r}\right) - (r-x)\sqrt{2rx - x^2}$$

Now, multiply the area by l (the length of the tank) to get the volume,

$$l\left[r^2 cos^{-1}\left(\frac{r-x}{r}\right) - (r-x)\sqrt{2rx - x^2}\right]$$

We know there are 231 in^3in a gallon. So we need to divide our volume by 231 to convert it to gallons.

$$\frac{l\left[r^2cos^{-1}\left(\frac{r-x}{r}\right)-(r-x)\sqrt{2rx-x^2}\right]}{231}$$

Hence, we can write as function using r (radius of the tank) = 18, x (depth of the tank), and l (length of the tank) = 48 as follows:

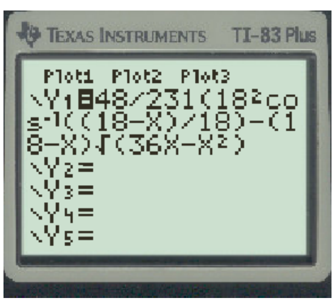
$$f(x)=\frac{48\left[18^2cos^{-1}\left(\frac{18-x}{18}\right)-(18-x)\sqrt{2\cdot 18\cdot x-x^2}\right]}{231}$$

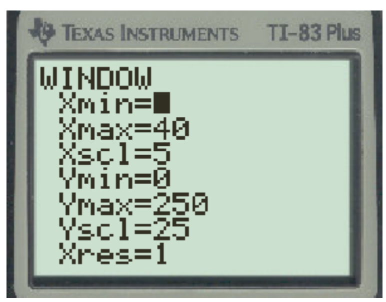
For our particular situation, since the superintendent determine he has 10” of oil in his tank. Thus, the volume of oil would be

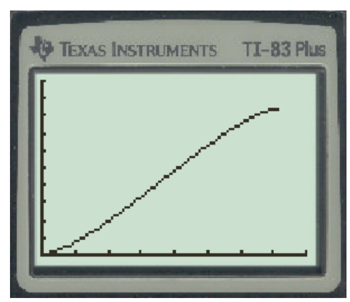
$$f(10)=\frac{48\left[18^2cos^{-1}\left(\frac{18-10}{18}\right)-(18-10)\sqrt{2\cdot 18\cdot 10-10^2}\right]}{231}=47.94\ gallons.$$

Now, can we use this equation to help the superintendent to use his dipstick to measure the amount of oil left in future? Let’s explore further.

The graph of $f(x)$ is below (using TI-83 Plus calculator):

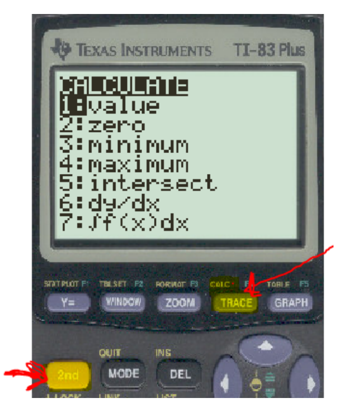


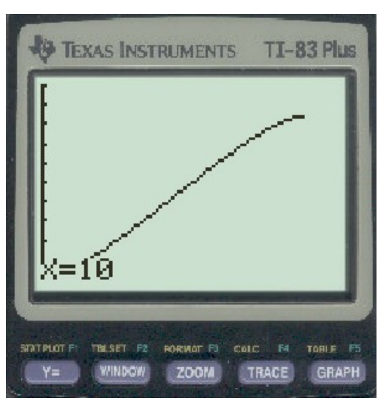


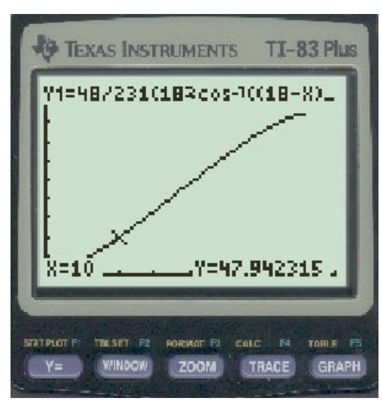


We can confirm our solution for $x=10$ using the calculator:

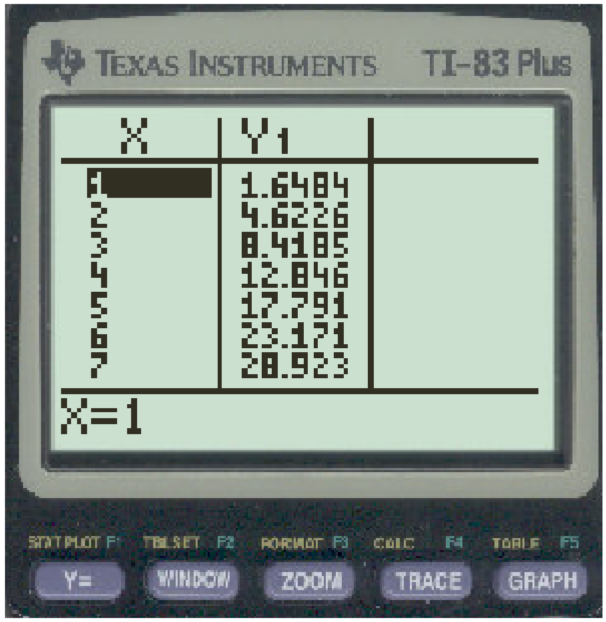
With $X=$ the length of measurement on dipstick, and $Y_1=$ Amount of Oil left on the tank



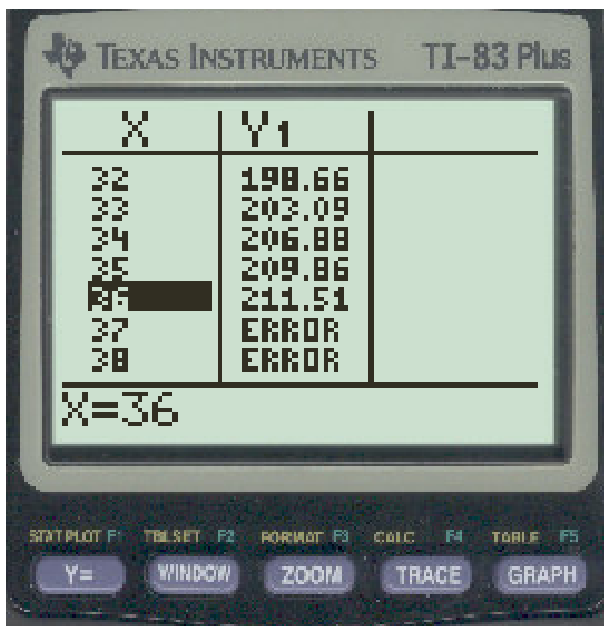




Hence the graph and the calculation on TI-83 confirm our earlier answer. Now, we can use the Table property of the TI-83 to calibrate a dip stick for this tank for future use:



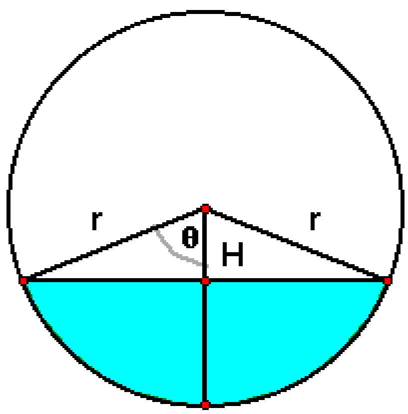
The table further reveals that, this particular tank has a maximum capacity of approximately 211.5 gallons.



Second Problem:

The owner of a marine farm grows oysters and mussels in barrel shaped buoys that lie on their sides. Each buoy supports about one ton of product. He bases the maturity of his product on its weight. The heavier the product, the lower the buoy sits in the water. But, lifting the buoys out of the water for weighing would disturb the growth process. The area of the submerged buoy cross-section (a section of a circle) divided by the total area of the cross-section (a circle shape) and then multiplied by 1000 kg will give him a good approximation of what he has growing. Given the radius, ***r****, and the perpendicular distance of the chord to the diameter,* ***H****, is there a way for this man to calculate the weight of his product without lifting the buoys out of the water?*

We can use the same concept developed earlier to solve this problem as well.



Again using Pythagorean formula, we can find the base of the triangle:

$$base_{triangle} = 2\sqrt{r^2 - H^2}$$

Now, the area of the triangle:

$$Area_{triangle} = \frac{1}{2} \times base \times height$$

$$= \frac{1}{2}\left(2\sqrt{r^2 - H^2}\right)H$$

$$= H\sqrt{r^2 - H^2}$$

Again, for the central angle,

$$\cos(\theta) = \frac{H}{r}$$

$$\theta = cos^{-1}\left(\frac{H}{r}\right)$$

Now, for our triangle, with central angle 2θ, we can form and solve the proportion as follows:

$$\frac{2\theta}{2\pi} = \frac{area_{sector}}{\pi r^2}$$

$$\frac{2cos^{-1}\left(\frac{H}{r}\right)}{2\pi} = \frac{area_{sector}}{\pi r^2}$$

$$area_{sector} = \frac{\pi r^2 \cdot 2cos^{-1}\left(\frac{H}{r}\right)}{2\pi}$$

$$area_{sector} = r^2 cos^{-1}\left(\frac{H}{r}\right)$$

So, our desired area,

$$r^2 cos^{-1}\left(\frac{H}{r}\right) - H\sqrt{r^2 - H^2}$$

Now, for approximation of the growth of the products, we need to divide this area by the total area of the cross section and multiply by 1000 kg:

$$weight\ of\ product = \frac{1000\left[r^2 cos^{-1}\left(\frac{H}{r}\right) - H\sqrt{r^2 - H^2}\right]}{\pi r^2}$$

Now, if we know the dimension of the buoys, we can solve the equation by substituting the values in above equation.